CASTELNUOVO–MUMFORD REGULARITY AND ARITHMETIC COHEN–MACAULAYNESS OF COMPLETE BIPARTITE SUBSPACE ARRANGEMENTS

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ABSTRACT. We give the Castelnuovo–Mumford regularity of arrangements of $(n-2)$-planes in $\mathbb{P}^n$ whose incidence graph is a sufficiently large complete bipartite graph, and determine when such arrangements are arithmetically Cohen–Macaulay.

1. Introduction

A subspace arrangement $\mathcal{A} = \{L_1, \dots, L_d\}$ is a finite collection of linear subspaces $L_i \subset \mathbb{P}^n$ with no inclusions $L_i \subset L_j$ for $i \neq j$. There are many relations among the algebraic properties of the defining ideal $I_\mathcal{A}$ of the arrangement, the combinatorial type of the arrangement, and the geometry of the arrangement itself. See for example the very recent survey article [8] on commutative algebra and subspace arrangements.

Following [1] we consider the **incidence graph** $\Gamma(\mathcal{A})$ of a subspace arrangement $\mathcal{A}$, defined as the graph with vertex set $\mathcal{A}$ and an edge XY for $X, Y \in \mathcal{A}$ if and only if the intersection $X \cap Y$ has greater than expected dimension. Thus, for example, for an arrangement of lines in $\mathbb{P}^3$, the incidence graph simply records which of the lines meet; for an arrangement of 2-planes in $\mathbb{P}^4$, the incidence graph records which planes meet along lines, and so on.

Plane arrangements whose incidence graph is a Petersen graph are studied in [1]. They are shown to link to surfaces with interesting geometric properties such as multisecant lines. The presence of a multisecant line intersecting a variety d times indicates a generator of degree at least d in the defining ideal of the variety, so the variety has Castelnuovo–Mumford regularity at least d. At the same time, for purposes of liaison theory it is natural to study whether a subspace arrangement is locally or even arithmetically Cohen–Macaulay.

Specifying $\Gamma(\mathcal{A})$ usually does not determine the Castelnuovo–Mumford regularity $\operatorname{reg} \mathcal{A}$, although it might bound it. For example, a line arrangement $\mathcal{A}$ in $\mathbb{P}^3$ with incidence graph a path of length of n can be constructed with hyperplane section realizing any given finite set of n points in $\mathbb{P}^2$ not all lying on a line. The regularity of $\mathcal{A}$ is equal to the regularity of its hyperplane section ([4, Prop. 20.20]), which is at most $n-1$ but depends on the position of the n points.

We show, however, that when $\mathcal{A}$ is an arrangement of $(n-2)$-planes in $\mathbb{P}^n$ and the incidence graph of $\mathcal{A}$ is a complete bipartite graph $K_{a,b}$ of type (a, b) then, for sufficiently large values of a, b, the regularity $\operatorname{reg} \mathcal{A}$ is uniquely determined.

An upper bound on $\operatorname{reg} \mathcal{A}$ is known. Indeed, Derksen and Sidman showed in [3] that if $\mathcal{A}$ is an arrangement of linear subspaces, then $\operatorname{reg} \mathcal{A} \leq |\mathcal{A}|$. Therefore, in the case where

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$\Gamma(\mathcal{A}) \cong K_{a,b}$, we have $\operatorname{reg} I_{\mathcal{A}} \leq a+b$. Even better, Giaimo [5] showed that for a reduced connected nondegenerate curve $C \subset \mathbb{P}^n$, $\operatorname{reg} C \leq \deg C - n + 2$ (this generalizes the case of an integral nondegenerate curve, treated in [6]). In our setting, if $\mathcal{A} \subset \mathbb{P}^3$ is a line arrangement with $\Gamma(\mathcal{A}) \cong K_{a,b}$, this gives $\operatorname{reg} \mathcal{A} \leq a+b-1$. Our main result shows that for most a, b, the actual regularity is lower than these upper bounds.

Theorem 1.1. *Let $\mathcal{A}$ be an arrangement of $(n-2)$-planes in $\mathbb{P}^n$ with incidence graph $\Gamma(\mathcal{A}) \cong K_{a,b}$, a complete bipartite graph of type (a,b). Suppose $a \leq b \leq 2$, $2 \leq a \leq b \leq 3$, or $3 \leq a \leq b$. Then the defining ideal $I_{\mathcal{A}}$ of the arrangement has regularity $\operatorname{reg}(I_{\mathcal{A}}) = \max(a+1, b)$.*

In addition we determine when these arrangements are arithmetically Cohen–Macaulay.

Theorem 1.2. *Let $\mathcal{A}$ be an arrangement of $(n-2)$-planes in $\mathbb{P}^n$ with $\Gamma(\mathcal{A}) \cong K_{a,b}$ where $a \leq b \leq 2$, $2 \leq a \leq b \leq 3$, or $3 \leq a \leq b$. Then $\mathcal{A}$ is arithmetically Cohen–Macaulay if and only if $b = a$ or $b = a+1$.*

For both theorems the omitted values are $a = 1$ and $b \geq 3$, or $a = 2$ and $b \geq 4$.

The idea is to reduce to the case of line arrangements, then use the special geometry of that setting.

We work over an algebraically closed field of any characteristic. We use $\mathcal{A}$ to denote both an arrangement (finite collection of subspaces) and the projective variety represented by that arrangement (the union of those subspaces).

2. Plane arrangements

For any arrangement $\mathcal{A}$ with $|\mathcal{A}| > 1$, the quotient by $\bigcap \mathcal{A}$ expresses $\mathcal{A}$ as a cone over an arrangement $\mathcal{B}$ in a possibly lower-dimensional space.

Lemma 2.1. *Let $\mathcal{A}$ be an arrangement of $(n-2)$-planes in $\mathbb{P}^n$. Suppose $\Gamma(\mathcal{A}) \cong K_{a,b}$ with $2 \leq a \leq b$. Then $\bigcap \mathcal{A}$ is an $(n-4)$-plane.*

Proof. By definition, there exist disjoint $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, $|\mathcal{A}_1| = a$, $|\mathcal{A}_2| = b$, and, for all $X, Y \in \mathcal{A}$,

- if $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$, then $\dim(X \cap Y) = n-3$, and
- if $X, Y \in \mathcal{A}_i$ for $i = 1, 2$, then $\dim(X \cap Y) = n-4$.

Let $X, Y \in \mathcal{A}_1$ be distinct and let $Z = X \cap Y$. Let $U \in \mathcal{A}_2$. Since $\dim X \cap Y = n-4$, $U \cap X$ and $U \cap Y$ must be distinct $(n-3)$-planes lying inside U, so $U \cap X$ and $U \cap Y$ intersect in some $(n-4)$-plane. This $(n-4)$-plane must be Z, so $Z \subset U$ and hence $Z \subset \bigcap \mathcal{A}_2$. On the other hand, as $|\mathcal{A}_2| \geq 2$ and any two subspaces in $\mathcal{A}_2$ intersect in an $(n-4)$-plane, $\bigcap \mathcal{A}_2$ has dimension at most $n-4$; thus $\bigcap \mathcal{A}_2 = Z = X \cap Y \supseteq \bigcap \mathcal{A}_1$.

By the same argument, $\bigcap \mathcal{A}_1 = U \cap V$ for any $U, V \in \bigcap \mathcal{A}_2$. It follows that $\bigcap \mathcal{A} = \bigcap \mathcal{A}_1 = \bigcap \mathcal{A}_2 = Z$, an $(n-4)$-plane. □

In the above situation, then, $\mathcal{A}$ is a cone over an arrangement $\mathcal{B}$ of lines in $\mathbb{P}^3$, with vertex an $(n-4)$-plane. We have that $\Gamma(\mathcal{B}) \cong \Gamma(\mathcal{A}) \cong K_{a,b}$. Indeed, if $X, Y \in \mathcal{A}$ correspond to lines $x, y \in \mathcal{B}$ (so that $X = x + \bigcap \mathcal{A}$, $Y = y + \bigcap \mathcal{A}$ as linear subspaces) then X and Y are adjacent in $\Gamma(\mathcal{A})$ if and only if x and y are adjacent in $\Gamma(\mathcal{B})$.

If $a \leq b = 2$ then again $\bigcap \mathcal{A}$ is an $(n-4)$-plane. (If $a = b = 2$ the above lemma applies. If $a = 1$, $b = 2$ then $\bigcap \mathcal{A} = \bigcap \mathcal{A}_2$ is an $(n-4)$-plane.) If $a = b = 1$ then $\bigcap \mathcal{A}$ is an $(n-3)$-plane and quotienting by any $(n-4)$-dimensional subspace expresses $\mathcal{A}$ as a cone over a

line arrangement in $\mathbb{P}^3$ consisting of two lines through a point. (While this line arrangement is again a cone and we could go down one dimension further, we choose to work with line arrangements.) This proves the following.

Lemma 2.2. *If $\mathcal{A}$ is an arrangement of $(n-2)$-planes in $\mathbb{P}^n$ and $\Gamma(\mathcal{A}) \cong K_{a,b}$ with $a \le b \le 2$ or $2 \le a \le b$ then $\mathcal{A}$ is a cone over an arrangement $\mathcal{B}$ of lines in $\mathbb{P}^3$ with $\Gamma(\mathcal{B}) \cong K_{a,b}$.*

Lemma 2.3. *If $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a complete bipartite arrangement of lines in $\mathbb{P}^3$, $\Gamma(\mathcal{A}) \cong K_{a,b}$ with $3 \le a \le b$ or $a \le b \le 3$, then there is a smooth quadric surface $Q \subset \mathbb{P}^3$ such that $\mathcal{A}$ lies on Q. Specifically, for each $i = 1, 2$, the lines of $\mathcal{A}_i$ lie in one of the two rulings of Q.*

Proof. Suppose $3 \le a \le b$. Let $X, Y, Z \in \mathcal{A}_1$ be distinct skew lines. There is a unique smooth quadric surface Q containing X, Y, Z as lines in one of its rulings. Then each line $L \in \mathcal{A}_2$ lies in Q, as it meets X, Y, Z, hence has three points in common with Q. Finally then each line $M \in \mathcal{A}_1$ lies in Q, as it meets each of the lines in $\mathcal{A}_2$, giving $b \ge 3$ points in common with Q.

That each line in $\mathcal{A}_1$ meets each line in $\mathcal{A}_2$ means they lie in opposite rulings of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$; that each pair of lines in $\mathcal{A}_1$ ($\mathcal{A}_2$) is skew means they lie in the same ruling.

A similar argument works if $a \le b = 3$. The claim is trivial if $a \le b \le 2$. □

3. Regularity

Recall the following definition.

Definition 3.1. The **(Castelnuovo-Mumford) regularity** of a sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is

$$\operatorname{reg} \mathcal{F} = \min\{d : H^i(\mathbb{P}^n, \mathcal{F}(d-i)) = 0 \, \forall i > 0\}.$$

For a subvariety $A \subset \mathbb{P}^n$ we denote by $\operatorname{reg} A$ the regularity $\operatorname{reg} I_A$ of the ideal sheaf of A. In particular, for an arrangement $\mathcal{A}$ we simply write $\operatorname{reg} \mathcal{A}$ for the regularity of the defining ideal of the arrangement.

For a comprehensive introduction to this topic, see for example [4, §20.5] or [7, §1.8].

Suppose $\mathcal{A}$ is an $(n-2)$-plane arrangement in $\mathbb{P}^n$ with $\Gamma(\mathcal{A}) \cong K_{a,b}$. As mentioned in the introduction, upper bounds on $\operatorname{reg} \mathcal{A}$ are known. We have $\operatorname{reg} \mathcal{A} \le a + b$ by a result of Derksen and Sidman [3], and indeed $\operatorname{reg} \mathcal{A} \le a + b - 1$ by a result of Giaimo [5], but these upper bounds, which do not take into account the special geometry of line arrangements lying on quadric surfaces, are not sharp.

General lower bounds for regularity seem to be less well known. If $\mathcal{A}$ is a line arrangement in $\mathbb{P}^3$ consisting of a lines in one ruling of a smooth quadric and b lines in the other ruling, then $\operatorname{reg} \mathcal{A} \ge \max\{a, b\}$. Indeed, a line on the quadric surface meets either a or b of the lines of $\mathcal{A}$. Therefore the defining ideal $\mathcal{I}_{\mathcal{A}}$ has a minimal generator in degree at least $\max\{a, b\}$. (We thank Jessica Sidman for pointing out to us this observation.) However, even this lower bound, taking into account the special geometry of $\mathcal{A}$, is still not sharp.

Theorem 3.2. *Suppose $\mathcal{A}$ is a line arrangement in $\mathbb{P}^3$ such that all lines in $\mathcal{A}$ lie in a smooth quadric surface Q and a lines of $\mathcal{A}$ lie in one of the rulings of Q, b lines of $\mathcal{A}$ lie in the other ruling, and $a \le b$. Then $\operatorname{reg} \mathcal{A} = \max\{a+1, b\}$.*

Proof. Let $\mathcal{I}_{\mathcal{A}} \subset \mathcal{O}_{\mathbb{P}^3}$ be the defining ideal sheaf of $\mathcal{A}$ in $\mathbb{P}^3$ and let $\mathcal{I}_{\mathcal{A},Q} \subset \mathcal{O}_Q$ be the defining ideal sheaf of $\mathcal{A}$ as a subvariety of Q. We have the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\cdot Q} \mathcal{I}_{\mathcal{A}} \longrightarrow \mathcal{I}_{\mathcal{A},Q} \to 0$$

By hypothesis, $\mathcal{I}_{\mathcal{A},Q} \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-a,-b)$. For $i \geq 1$ and $d \geq i-1$, $H^i(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-i-2)) = 0$, so

$$\begin{aligned} H^i(\mathbb{P}^3, \mathcal{I}_{\mathcal{A}}(d-i)) &\cong H^i(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d-i-a, d-i-b)) \\ &\cong \bigoplus_{j+k=i} H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-i-a)) \otimes H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-i-b)) \end{aligned}$$

Now $H^1(\mathbb{P}^3, \mathcal{I}_{\mathcal{A}}(d-1)) = 0$ if and only if $d-1-a < 0$ or $d-1-b > -2$, and $d-1-a > -2$ or $d-1-b < 0$; this is equivalent to $d \leq a$ or $d \geq b$. And $H^2(\mathbb{P}^3, \mathcal{I}_{\mathcal{A}}(d-2)) = 0$ if and only if $d-2-a > -2$ or $d-2-b > -2$; since $a \leq b$, this is equivalent to $d \geq a+1$. $\square$

Note that in the case $a = b$, $\mathcal{A}$ is a complete intersection of type $(2, a)$, which already implies $\operatorname{reg}\mathcal{A} = a+1$ [7, Example 1.8.27].

This result generalizes to higher dimensions, yielding the statement given in the Introduction.

Proof of Theorem 1.1. Suppose $a \leq b \leq 2$ or $2 \leq a \leq b \leq 3$ or $3 \leq a \leq b$. By Lemma 2.2 $\mathcal{A}$ is a cone over a line arrangement $\mathcal{B}$ in $\mathbb{P}^3$ with $\Gamma(\mathcal{B}) \cong K_{a,b}$. By Lemma 2.3 $\mathcal{B}$ lies on a quadric, thus satisfies the hypotheses of Theorem 3.2. Hence $\operatorname{reg}\mathcal{B} = \max\{a+1, b\}$. Since $\mathcal{A}$ is a cone over $\mathcal{B}$, we have $\operatorname{reg}\mathcal{A} = \operatorname{reg}\mathcal{B}$. (See, for example, [4, Prop. 20.20]: the regularity of $\mathcal{A}$ is equal to the regularity of its linear section $\mathcal{B}$.) $\square$

Example 3.3. If X, Y are projective varieties such that $X \cap Y$ is zero-dimensional, then $\operatorname{reg} X \cup Y \leq \operatorname{reg} X + \operatorname{reg} Y$, by a result of Caviglia (Corollary 3.4 in [2]). We can easily give examples in which equality occurs. Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^3$ with $\Gamma(\mathcal{A}) \cong K_{a_1,a_2}$, lying on a smooth quadric surface Q. Let $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$, where $\Gamma(\mathcal{B}) \cong K_{b_1,b_2}$, $\Gamma(\mathcal{C}) \cong K_{c_1,c_2}$, $a_i = b_i + c_i$ for $i = 1, 2$. Theorem 3.2 applies to $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ since they lie on Q; and the intersection $\mathcal{B} \cap \mathcal{C}$ is zero dimensional. Then $\operatorname{reg}\mathcal{A} = \operatorname{reg}\mathcal{B} + \operatorname{reg}\mathcal{C}$ if and only if one of the following cases occurs: $b_1 > b_2$ and $c_1 > c_2$ (so $\operatorname{reg}\mathcal{A} = a_1 = b_1 + c_1 = \operatorname{reg}\mathcal{B} + \operatorname{reg}\mathcal{C}$); or similarly $b_1 < b_2$ and $c_1 < c_2$; or $b_1 = b_2 + 1$ and $c_2 = c_1 + 1$ (so $a_1 = a_2$, $\operatorname{reg}\mathcal{A} = a_1 + 1 = b_1 + c_2 = \operatorname{reg}\mathcal{B} + \operatorname{reg}\mathcal{C}$); or similarly $b_2 = b_1 + 1$ and $c_1 = c_2 + 1$.

Cones over these arrangements give examples for which $\operatorname{reg}\mathcal{B} \cup \mathcal{C} = \operatorname{reg}\mathcal{B} + \operatorname{reg}\mathcal{C}$ while $\mathcal{B} \cap \mathcal{C}$ is positive-dimensional.

We can also now prove the condition given in the Introduction determining when these arrangements are arithmetically Cohen–Macaulay.

Proof of Theorem 1.2. First suppose $\mathcal{A}$ is a line arrangement in $\mathbb{P}^3$ lying on a smooth quadric surface, with $\Gamma(\mathcal{A}) \cong K_{a,b}$, $a \leq b$. The computation in the proof of Theorem 3.2 shows that if $a < d < b$, then $H^1(\mathbb{P}^3, \mathcal{I}_{\mathcal{A}}(d-1)) \neq 0$. Thus if $b \geq a+2$ ($\mathcal{A}$ is "unbalanced"), then $\mathcal{A}$ is not projectively normal and not arithmetically Cohen–Macaulay.

Conversely, if $b = a$ or $b = a+1$ then the same computation shows $H^1(\mathbb{P}^3, \mathcal{I}_{\mathcal{A}}(d-1)) = 0$ for all d, so $\mathcal{A}$ is arithmetically Cohen–Macaulay.

In higher dimensions, if $\mathcal{A}$ is an $(n-2)$-arrangement in $\mathbb{P}^n$ with $\Gamma(\mathcal{A}) \cong K_{a,b}$ with $a \leq b \leq 2$, $2 \leq a \leq b \leq 3$, or $3 \leq a \leq b$, then $\mathcal{A}$ is a cone over a line arrangement $\mathcal{B} \subset \mathbb{P}^3$ lying on a quadric with $\Gamma(\mathcal{B}) \cong K_{a,b}$, and so $\mathcal{A}$ is arithmetically Cohen–Macaulay if and only if $\mathcal{B}$ is, if and only if $b = a$ or $b = a+1$. $\square$

Remark 3.4. We can do slightly better in $\mathbb{P}^3$. If $\mathcal{A}$ is a line arrangement in $\mathbb{P}^3$ with $\Gamma(\mathcal{A}) \cong K_{a,b}$ where $a \leq b \leq 3$ or $3 \leq a \leq b$, then $\mathcal{A}$ lies on a quadric surface (Lemma 2.3), so

$\operatorname{reg} \mathcal{A} = \max\{a+1, b\}$ (Theorem 3.2) and $\mathcal{A}$ is arithmetically Cohen–Macaulay if and only if $b = a$ or $b = a + 1$ (same proof as Theorem 1.2).

This simply adds the case $(a, b) = (1, 3)$ to the list of cases already given in Theorems 1.1 and 1.2.

References

1. Hirotachi Abo, Holger P. Kley, and Chris Peterson, *Petersen plane arrangements and a surface with multiple 7-secants*, Adv. Geom. **9** (2009), no. 3, 349–369.
2. Giulio Caviglia, *Bounds on the Castelnuovo-Mumford regularity of tensor products*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 1949–1957 (electronic).
3. Harm Derksen and Jessica Sidman, *A sharp bound for the Castelnuovo-Mumford regularity of subspace arrangements*, Adv. Math. **172** (2002), no. 2, 151–157.
4. David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.
5. Daniel Giaimo, *On the Castelnuovo-Mumford regularity of connected curves*, Trans. Amer. Math. Soc. **358** (2006), no. 1, 267–284 (electronic).
6. L. Gruson, R. Lazarsfeld, and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), no. 3, 491–506.
7. Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik., vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.
8. Hal Schenck and Jessica Sidman, *Commutative algebra of subspace and hyperplane arrangements*, 2012.

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